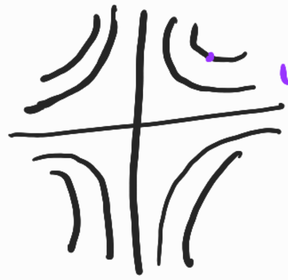
Main Theorem (Chaudhuri-M, 24).

X be a klt 3-fold, $\mathcal{F}$ be a corank one foliation on X with log canonical sing. Then any $K_{\mathcal{F}}$-MMP exists and terminates.

What is a foliation ??

Consider $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \in T_{\mathbb{C}^2} \Rightarrow$  ✓

$(\dot{x}(t) = x, \dot{y}(t) = -y)$

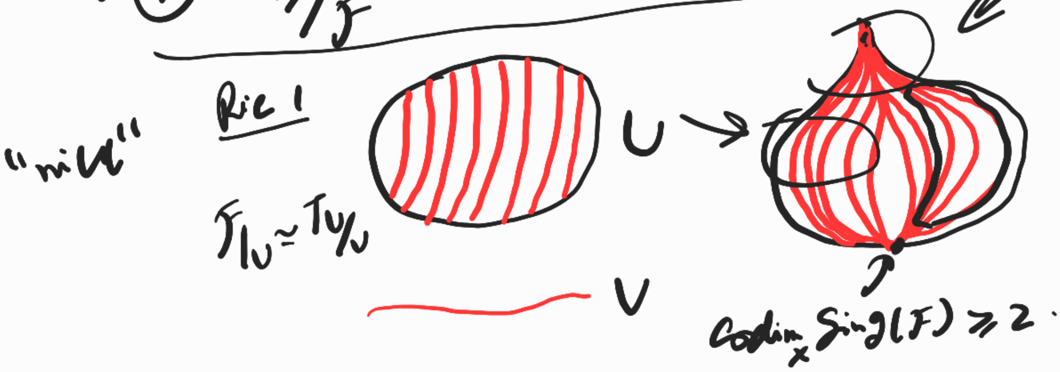
Can we extend such structure to Projective varieties ?? any - FOLIATION !!

Def:- Let X be normal proj var and let $\mathcal{F} \subset T_X$ be a coherent subsheaf which satisfies

✓ (i) $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$

✓ (ii) $T_X/\mathcal{F}$ is torsion free.

Descriptive Picture.



Why do we care / what do we want ??

Foliations naturally pop up in many diff context.

1) fibration $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$ naturally defines a foliation on X.

ii) $\mathcal{F}^{\vee\vee} \subset T_X$, Destabilizing subsheaves of T_X. (used in the proof of Abundance dim 3).

iii) It also pops up in the proof the B-B decomposition theorem for kalt varieties.

We would like to group these structure together in a sensible manner

(Construct Proper Moduli following the approach of KSBA, ... HMX)

Important ingredient $\longrightarrow$ MMP

Even for the case of $\mathcal{F} = T_X$, we need to consider mild sing. Motivated by that we

First introduce the "mild foliated" sing.

Def: $(Y, \mathcal{F}_Y) \xrightarrow{\pi} (X, \mathcal{F}_X)$ be a bir morphism
we can write

$$K_{\mathcal{F}_Y} = \pi^* K_{\mathcal{F}_X} + \sum a(E_i, \mathcal{F}_X) E_i \quad {}^{**}$$

we say $(X, \mathcal{F}_X)$ has lc (resp canonical) if
$a(E_i, \mathcal{F}_X) \geq -\varepsilon(E_i)$ (resp $a(E_i, \mathcal{F}_X) \geq 0$) $\forall \pi$.

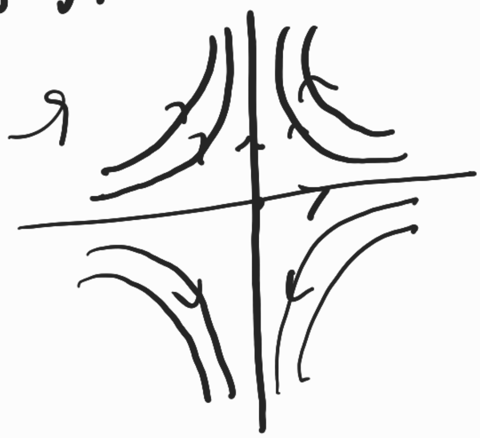
$$\varepsilon(E_i) = \begin{cases} 0 & \text{if } E_i \text{ is } \mathcal{F}_Y\text{-inv} \\ 1 & \text{o.w.} \end{cases}$$

(union of leaves away from Sing $\mathcal{F}$)

dim X − rank F.

Another V. Important class of Sing (Corank $\mathcal{F} = 1$).

F-dlt Sing: Roughly the separatrices meet in a SNC manner along lc center (Sing $\mathcal{F}$) of $\mathcal{F}$. (and variety is smooth at those points).



$\langle x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \rangle \in T_{\mathbb{C}^2}$

F-dlt foliation.

Prototype

Note: $U = \text{Sm}\,\mathcal{F}$, $\Omega^1_{\mathcal{F}}|_U$ is v.b $\Rightarrow$ extending $\wedge^r(\Omega^1_{\mathcal{F}}|_U)$ we get $\mathcal{O}_X(K_{\mathcal{F}})$, a divisorial shf.

Conj: $(X,\mathcal{F})$ foliated variety with $\mathcal{F}$ having lc sing. Then the $K_{\mathcal{F}}$-MMP exists and terminates ✓

Naive Take 1

Replace K_X with $K_{\mathcal{F}}$ and apply same techniques. ✗

$K_{\mathcal{F}}$ nef $\not\Rightarrow$ semi ample.

1) Base Point free theorem is false for $K_{\mathcal{F}}$.

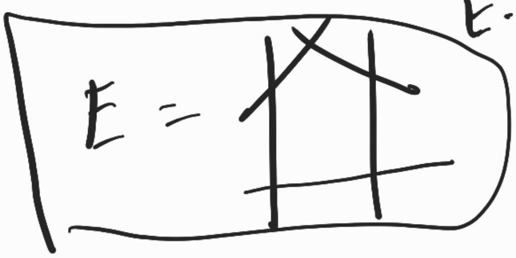
$$\begin{array}{ccc} (Y,\mathcal{F}') & \longrightarrow & \\ \downarrow & & \\ (X,\mathcal{F}) & \longrightarrow & \mathbb{D}\times\mathbb{D}/\Gamma \end{array}$$

$K_{\mathcal{F}}$, E.L.L.

$K_{\mathcal{F}}|_E$ is non-torsion.

$\Gamma \subset \mathrm{Aut}(\mathbb{D}\times\mathbb{D})$

$\mathbb{D}\times\mathbb{P}$
$\downarrow$
$\mathbb{P}$

$E =$ 

② K-V. Vanishing is false

In the case of $\dim X = 2$, Conj is true.
(Brunella, McQuillan, Mendes).

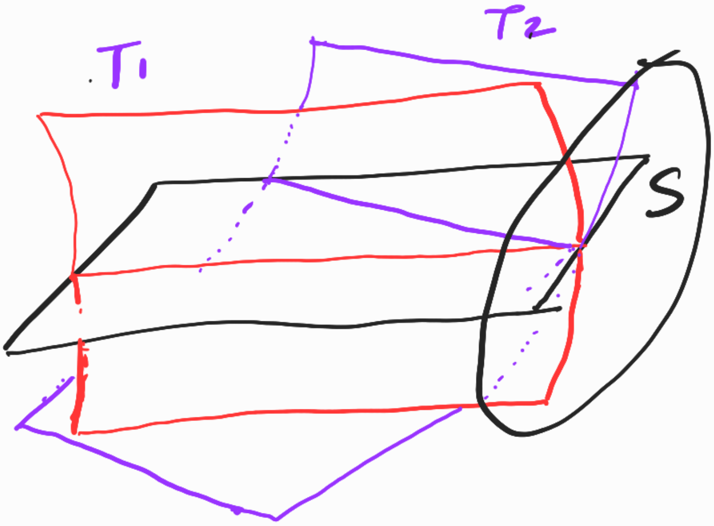
In case of $\dim X = 3$, corank $\mathcal{F} = 1$, with $\mathcal{F}$-dlt (non-dicritical) singularity
[Cascini-Spicer'21, Spicer-Svaldi'22].

Their strategy

Contraction Thm : CHR is not big ⟹ Classical bend and break

In a small analytic/formal nbd of some "nice" invariant divisor, and replace a $K_{\mathcal{F}}$-problem to $(K_{\hat{X}} + \Sigma T_i)|_S$ problem + Adjunction. $(\hat{X}, \Sigma T_i) \to lc$

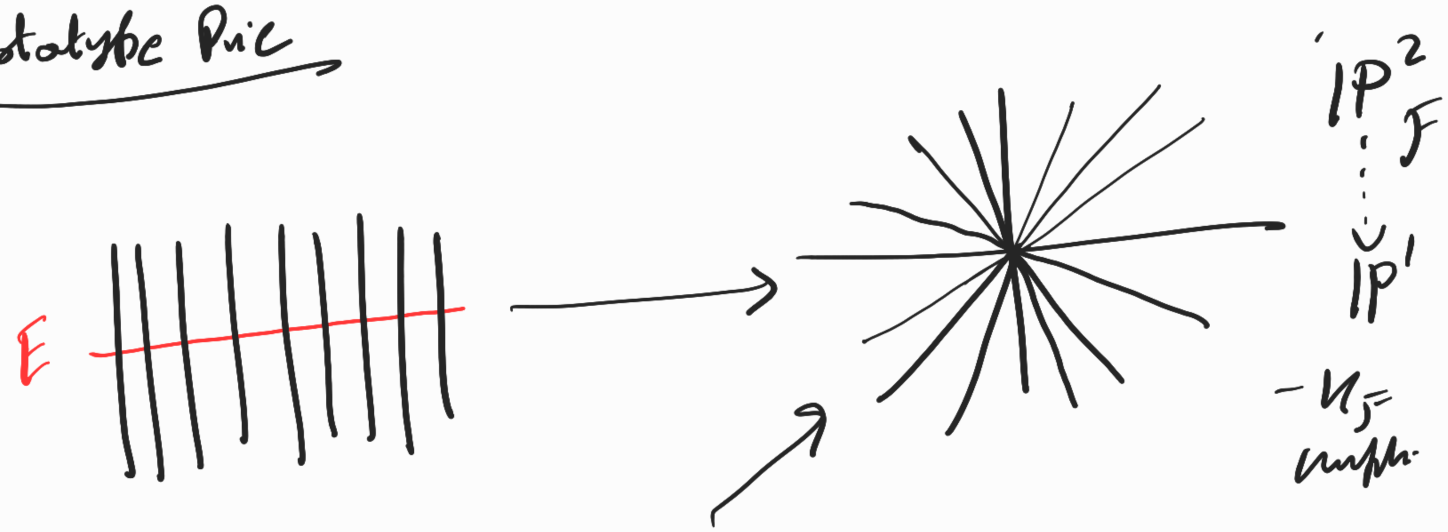
Flips - Working in an analytic nbd of curves we find Δ such that $K_{\mathcal{F}} \cdot C = (K_X + \Sigma T_i + \Delta) \cdot C$
⟹ Construct the flip in an etalé nbd.



[dim X = 3
corank F = 1]

For strictly log canonical sing of $(X, \mathcal{F}_X)$, if $(Y, \mathcal{F}_Y) \xrightarrow{\pi} (X, \mathcal{F}_X)$ is a log resolution (exists due to Cano), $\exists$ π-exceptional E, which is $\mathcal{F}_Y$-non invariant. [In particular $a(E, \mathcal{F}) = -\varepsilon(E) = -1$.]

Prototype Pic



These are very natural class of foliations for example Fano foliations ($-K_{\mathcal{F}}$ ample) with lc sing are always strictly lc. (AD)

(Chaudhuri - M, Zu)
MMP exists and terminates for lc foliations on all 3-fold. (even generalized foliated pairs!!)

Main Challenges

1) Contraction THM

CS21 heavily relies on the fact that finitely many T_i's meet $\mathrm{Sing}\,\mathcal{F}$.
As in our prototype, that's not the case for us.

Our Strategy.

$R \in \overline{NE}(X)$ extremal, $K_{\mathcal{F}} \cdot R < 0$, H_R supporting Cartier, big.

$H_R = K_{\mathcal{F}} + \Delta + A$.

$(X, \mathcal{F}) \longrightarrow$ go to $(\bar{X}, \bar{\mathcal{F}}, \bar{\Delta}) \xrightarrow{\pi} (X, \mathcal{F})$.

F-dlt resolution

$\Big\downarrow$ we may

run a $K_{\bar{\mathcal{F}}} + \bar{\Delta}$ - MMP

with scaling of $\pi^* A$ to contract Null$(\pi^* H_R)$.

fail to do so as

if $\exists E_i$ transverse to the foliation.

We stop when scaling $\lambda_i < 1$.

let the ith step be $\bar{X}_i = \bar{X}'$ $\quad \lambda_i = \lambda'$

Construct an adjoint div

$$K = (K_{\bar{\mathcal{F}}} + \bar{\Delta} + \lambda' \bar{A}) + \delta (K_{\bar{X}} + E).$$

$\bar{\Delta} = \sum \varepsilon(E_i) E_i$

s.t. ϕ is K-neg and K is big.

now we run a $\varphi_* K = K'$ - MMP

$\Rightarrow (K_{\bar{\mathcal{F}}} + \bar{\Delta} + \lambda' \bar{A})$ - trivial

and $K_{\bar{X}} + E$ - neg.

$\varphi \circ \psi$ is $\pi^* H_R$-trivial and contracts Null$(\pi^* H_R)$ (using classical BPFT for $\frac{1}{3}K''$)

thus proving H_R semiample and done.

Flips - are exactly same, constant relative canonical model of Z, by recall.

More interestingly

In rank one case, a flipping curve cannot be a strictly lc center / cannot pass through.

We produce an example s.t. flipping curve is a strictly lc center of the foliation using toric foliation.

$$\begin{tikzcd}
\mathbb{P}^1 & & \bar{X} \arrow[ll] \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1 \arrow[r, dotted, hook] & X \arrow[dr] & & X^+ \arrow[dl] \\
& & Z &
\end{tikzcd}$$

locus of indeterminacy flipped curve

X